THE KIRILLOV MODEL OF $\mathrm{GL}_2(F)$ FOR NONARCHIMEDEAN FIELDS F

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1. Introduction

In these notes, we shall prove the existence and uniqueness of the Kirillov Model for admissible representations of $\mathrm{GL}_2(F)$ where F is a nonarchimedean field. We follow, for the most part, Godement's notes [3], with occasional additions from the original Jacquet-Langlands opus [4]. For a treatment of the finite-dimensional representations of $\mathrm{GL}_2(\mathbb{F}_q)$ along the lines of what we do below for a local field F (i.e., the "harmonic analysis approach"), we recommend Piatetski-Shapiro's book [5]; the algebro-geometric approach to the finite field case can be found in [1] or [2]. We will, at times, compare the development given below to the finite-field case as presented in [5].

2. Admissible Representations

Throughout this section, we let F be a nonarchimedean, locally compact field. We let $\mathcal{O}_F$ be the associated ring of integers, and $\mathcal{O}_F^\times$ the group of units of $\mathcal{O}_F$. We fix once and for all a nontrivial, continuous, additive character, $\tau : F \to \mathbb{C}^1$, with $\mathbb{C}^1$ the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$; we have $\tau(x+y) = \tau(x)\tau(y)$. The prime ideal of $\mathcal{O}_F$ will be denoted by $\mathfrak{p}$; we pick a uniformizer ϖ of $\mathfrak{p}$. We denote the associated valuation on F by $v_\mathfrak{p}$. Let $q = |\varpi|_F^{-1}$, the cardinality of the residue field $\mathcal{O}_F/\mathfrak{p}$. We note that q is a power of p, a prime integer. The kernel of τ will be an open subgroup of F; i.e., $\mathfrak{p}^{-d}$ for some integer d. From now on d will refer to this specific integer.

We will frequently deal with *multiplicative* characters χ, which will generally mean continuous maps $\chi : \mathcal{O}_F^\times \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$. (Sometimes, however, we will use χ to denote multiplicative character on all of $F^\times$; we note that such χ is uniquely determined by the value of $\chi(\varpi)$ and the restriction of χ to $\mathcal{O}_F^\times$.) We define the **conductor** of a character χ of $\mathcal{O}_F^\times$ to be the smallest value of f such that χ is constant on $1 + \mathfrak{p}^f$ (i.e., the *largest* $\mathfrak{p}^f$ such that χ restricted to $1+\mathfrak{p}^f$ is trivial). Note that unlike d defined above, which can be any integer, positive or negative, f can only be $0, 1, \ldots$. If, moreover, $f = 0$, then the character χ is trivial. Observe that the set $1 + \mathfrak{p}^f$ is the connected component of the identity of the kernel of χ, and is (as we will show below), itself a multiplicative subgroup of $F^\times$ for all f. We note that Godement [3] and Jacquet-Langlands [4] use the term "conductor" to refer to the ideal $\mathfrak{p}^f$ as opposed to the integer f. This may sow some confusion among those partial to Godement and Jacquet-Langlands's nomenclature: when we refer to a character with a "large" conductor, we will mean a character with a large numerical value of f; of course,

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however, this will mean that the *ideal* $\mathfrak{p}^f$ is *small*. We hope that our readers will forgive us for this departure from the scriptures. Finally, we note that there are only finitely many characters with conductor bounded by a given constant.

We let $\mathscr{S}(F)$ denote the complex vector space of locally-constant, compactly-supported functions from F to $\mathbb{C}$. We now pick an additive Haar measure dx on F: we note that $d(ax) = |a|\,dx$. The measure dx is unique up to a multiplicative constant; however, we may specify dx uniquely by insisting that dx is "self-dual," i.e., that the Fourier inversion formula can be written:

$$\widehat{f}(y) = \int_F f(x)\overline{\tau(xy)}dx \implies f(x) = \int_F \widehat{f}(y)\tau(xy)dy$$

for all $f \in \mathscr{S}(F)$. We shall denote a multiplicative Haar measure on F^* by d^*x; we will generally pick the unique measure such that $\int_{\mathcal{O}_F^\times} d^*x = 1$. (Recall that $\mathcal{O}_F^\times$ is the maximal compact of $F^\times$.) Note that $d^*x = c \cdot \frac{dx}{|x|}$ with c a constant. However, we should not expect this constant to be, in general, 1. For example, if $d = 0$ – i.e., $\ker(\tau) = \mathfrak{p}^0 = \mathcal{O}_F$ – then applying Fourier inversion to the test function $f = 1_{\mathcal{O}_F}$ (i.e., the indicator function of $\mathcal{O}_F$), we see that $\widehat{f}(y) = 1_{\mathcal{O}_F}$, too. Thus Fourier inversion holds iff $\mathrm{Vol}(\mathcal{O}_F) = 1$. However, in this case, the measure $\frac{dx}{|x|}$ assigns a volume of $\frac{q-1}{q}$ to $\mathcal{O}_F^\times$ where q is the cardinality of the residue field $\mathcal{O}_F/\mathfrak{p}$. Thus we have $d^*x = \frac{q}{q-1} \cdot \frac{dx}{|x|}$.

Finally let $G_F := \mathrm{GL}_2(F)$, sometimes just written G; and let $M_F = \mathrm{GL}_2(\mathcal{O}_F)$, sometimes just written M. Note that M is a maximal compact subgroup of G . We shall pick the unique Haar Measure dg on G such that M has volume 1.

In contrast to the case of $\mathrm{GL}_2(\mathbb{F}_q)$, the finite-dimensional representations of G_F for nonarchimedean F will all be rather uninteresting (see Lemma 3.1). Recall that the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ of dimension greater than 1 have dimension $q-1$, q, and $q+1$; those of dimension $q - 1 = |\mathbb{F}_q^\times|$ are called the "cuspidal" representations, those of dimension q are called the "special" representations, and those of dimension $q+1$ are called the "principle series." We see that the dimensions of non-character irreducible representations grows with the cardinality of the field. Thus we might expect that the only irreducible representations of dimension greater than one for $\mathrm{GL}_2(F)$ with (an infinite) nonarchimedean field F will be infinite-dimensional.[1] To tame the potentially wild behavior of such infinite-dimensional representations, we will restrict our attention to a very special class of infinite-dimensional representations called *admissible* representations. These will bear a strong resemblance to the finite-dimensional representations of $\mathrm{GL}_2(\mathbb{F}_q)$ classified in [5]. We shall now define this notion precisely.

Let π be a representation of G_F on some complex vector space (generally infinite-dimensional) V. If $v \in V$ is some vector, we let $\mathrm{Stab}_G(v)$ be the stabilizer of v in G, i.e., $\{g \in G_F : \pi(g)(v) = v\}$. We call the representation π **smooth** if $\mathrm{Stab}_G(v)$ is an open

[1] This intuition is somewhat imprecise, however, since $F_p = \overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$ – it does not equal $\mathbb{Q}_p$. The claim made, however – that there are no finite-dimensional representations of $\mathrm{GL}_2(F)$ for F nonarchimedean, is in fact true (Prop 3.1 a)), so there may be something to it. A more persuasive heuristic, however, is that the explicit constructions of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in [5] are on certain function spaces which we would expect to be infinite-dimensional if the $|F|$ is infinite.

subgroup of G_F for each v. We say that π is **admissible** if, given $H \subset G$ an open subgroup, the set of H-stable vectors $\{v \in V : \forall h \in H, \pi(h)(v) = v\}$ (which forms a vector subspace of V) is finite-dimensional.

Another equivalent classification is given as follows. We may restrict π as a representation of G to the maximal compact $M = \mathrm{GL}_2(\mathcal{O}_F)$. The condition of the admissibility of π is equivalent to V decomposing as a direct sum

$$V \simeq \bigoplus_{\theta \in \mathrm{Irr} M} \theta^{n_\theta}, \tag{1}$$

where θ ranges over all non-isomorphic finite-dimensional irreducible representations of M, and $n_\theta < \infty$ for all θ. I.e., the multiplicity of each irreducible, finite-dimensional representation of θ of M in V is finite, and the direct sum of all such irreducible representations of M equals V.

Hence, while admissible representations in general have infinite dimension, this dimension is, in fact, countable. This is nice from the perspective of algebra but, perhaps, a bit odd from the perspective of functional analysis; in particular, it is a consequence of the Baire category theorem that the vector space V cannot be a Banach or Hilbert space. However, this should not vex us: we may always complete V with respect to a norm if we are in possession of one; and, in any case, when we construct models for V as a space of functions (from, say $F^\times$ or G to $\mathbb{C}$), the functions representing vectors of V will all be locally constant, and so have a very much more discrete flavor than those we are familiar with from usual (Archimedean) functional analysis.

We recall from representation theory that for every representation π of G on a vector space V, there is a canonical dual representation of G on V^* given by $g \mapsto \pi(g^{-1})^t$, and we recall from linear algebra that the dual vector space of a direct sum of is a direct product. Thus we see that

$$V^* \simeq \prod_{\theta \in \mathrm{Irr} M} (\theta^*)^{n_\theta},$$

which no longer has countable dimension and so cannot be admissible. We can remedy this situation by considering instead the smaller space:

$$\widetilde{V} :\simeq \bigoplus_{\theta \in \mathrm{Irr} M} (\theta^*)^{n_\theta}.$$

We note that this corresponds to the subspace of V^* for which $v^* \in V^*$ is invariant under some open subgroup of G. Thus $\widetilde{V}$ is, in fact, a subrepresentation of the dual representation, which is admissible. We call this the **contragradient** representation of π, denoted by $\widetilde{\pi}$.

If V_1 is a G-subrepresentation of π then $V_1^\perp := \{v^* \in \widetilde{V} : \forall v \in V_1,\ v^*(v) = 0\}$; i.e., the set of all $v^* \in \widetilde{V}$ annihilating V_1, is a G-subrepresentation of $\widetilde{\pi}$ on $\widetilde{V}$, and $(V_1^\perp)^\perp = V_1$. If V has no nontrivial invariant subspaces then we say it is **irreducible**. Our purpose here is to provide a concrete model for every irreducible, admissible representations of G_F.

We shall find that this classification beautifully parallels the classification of irreducible finite-dimensional complex representations of $\mathrm{GL}_2(\mathbb{F}_q)$. This will offer some *ex post* confirmation that admissible representations of G_F are the "right" analogue of the finite-dimensional irreducible representations of the finite-field case.

But perhaps the main theoretical virtue of admissible representations is that they satisfy an analogue of Schur's Lemma. Indeed: if $T \in \mathrm{End}(V)$ is an operator commuting with π, then it preserves each θ^{n_θ}; since these are finite dimensional, T has an eigenvector in each such θ^{n_θ}. As in the usual Schur lemma, we now consider a single eigenvalue λ of T; and let V' be the kernel of $T - \lambda I$. We see that V' is a nontrivial subrepresentation of V, so by irreducibility it is equal to V. Thus $T = \lambda I$ is a scalar.

Note, however, that we do *not* have semisimplicity of admissible G_F-representations.[2] I.e., given a $\pi(G_F)$-invariant subspace $V' \subset V$, we may not in general be able to find a $\pi(G_F)$-invariant complement of V' in V; what we do have is a canonical correspondence between the invariant subspaces of V and the invariant subspaces of the contragradient $\widetilde{V}$ as described above.

3. The Kirillov Model: First Steps

We shall construct, for any irreducible admissible representation π of G_F, a model for π whose underlying vector space is a particular class of locally constant functions on $F^\times$. First, however, we shall show that the finite-dimensional representations of G are not terribly interesting:

Proposition 3.1 (c.f., Prop 2.7 of [4])**.** *Let π be an irreducible admissible representation of G_F on the vector space V.*

a) If V is finite dimensional, then $\dim V = 1$, *and there exists a character*[3] *χ of $F^\times$ such that*

$$\pi(g) = \chi(\det g).$$

b) If V is infinite-dimensional there is no nonzero vector invariant under the action of the unipotent subgroup $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

(We note that part b) indicates a departure from the finite field case: in the finite field case, [5] defines the Jacquet module as the vector space unipotent invariants. This has dimension 0, 1 or 2, corresponding to the representation being cuspidal, special, or principle series, respectively. In the case of nonarchimedean local fields F, we see the space of unipotent invariants is an uninteresting structure: it is always 0. For nonarchimedean F one rather

[2] Note that this is *not* a problem for the maximal compact M, however, as we can apply the usual averaging trick for compact groups to find an M_F-invariant Hermitian form on V, and then take the orthocomplement of V'. Thus V is semisimple as an M_F-representation: indeed, this follows from (1).

[3] Here we depart from the language of Jacquet-Langlands [4] slightly. For us a "character" refers simply to a continuous (a.k.a., locally constant) multiplicative homomorphism $\chi : F^\times \to \mathbb{C}^\times$. Jacuquet-Langlands calls these "quasicharacters" and reserves the title of "character" for those $\chi : F^\times \to \mathbb{C}^\times$ which take their values in the complex unit circle $\mathbb{C}^1$. We shall call these "unitary characters."

define the Jacquet module as a *quotient space* of V on which the unipotent elements acts trivially, rather than as a *subspace* of V.)

Proof. a) If π is finite-dimensional then its kernel H is an open normal subgroup of G_F. The idea is to consider the intersection of H with the unipotent subgroup. We see that there must exist an $\varepsilon > 0$ such that

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in H$$

for all $|y| < \varepsilon$. For any $x \in F$, we may find an $a \in F^\times$ such that $|ax| < \varepsilon$. We have the identity[4]

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{2}$$

which implies, by normality of H, that all unipotent matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H$. But an identical argument reveals that the "opposite" uinpotent matrices, i.e., those matrices of the form $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, lie in H for all $x \in F$.

Since the unipotent and opposite-unipotent matrices together generate $\mathrm{SL}_2(F)$, we see that H contains $\mathrm{SL}(2, F)$. Thus π must factor through the determinant, and we see that $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$ for all $g_1, g_2 \in G_F$. Thus, by Schur's Lemma, every $\pi(g)$ is a scalar multiple of the identity:

$$\pi(g) = \chi(\det g)I$$

for some multiplicative homomorphism $\chi : F^\times \to \mathbb{C}^\times$. The continuity of χ follows from smoothness of π and noting that

$$\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \chi(a)I.$$

This completes the proof of a).[5]

[4] Warning: there is an egregious typo in this identity in the online version of Jacquet-Langalnds [4].

[5] One might wonder why this argument fails if, say, rather than G_F we consider, $M_F = \mathrm{GL}_2(\mathcal{O}_F)$. Indeed, we know that M_F must have *many* finite-dimensional representations, and so it must in turn have many normal open subgroups. The problem is that the matrices with which we conjugated our unipotent elements, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, do not lie in M_F. We will use the fact that the subgroup of matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ conjugates unipotent matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ by scaling the entry x, i.e., identity (2), several times, so it is worth internalizing this algebraic identity.

b) Say that $v \in V$ is fixed by $\pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for all $x \in F$. Let H_0 be stabilizer of v, and let H be the set of matrices that preserve the 1-dimensional subspace $\mathbb{C}v$ (i.e., $\{g \in G_F : \pi(g)v = \lambda v \text{ for some } \lambda \in \mathbb{C}\}$). We will show that H has finite index in G_F. This in turn will imply that $\pi(G_F)(v)$ is a finite-dimensional subrepresentation of V (indeed, it has basis $\pi(g_i)v$ where g_i ranges over the finite collection of coset representatives for G_F/H); thus, $\pi(G_F)(v) = V$ by irreducibility. But this contradicts the assumption of infinite-dimensionality of V, so we are done.

So we must prove that $H \subset G$ has finite index. To this end, we first prove that $H_0 \supset \mathrm{SL}_2(F)$. We already have that H_0 contains all unipotent matrices; like in a), we would like to show that it contains all opposite unipotent matrices. By smoothness of π, we know that H_0 is open, whence it must contain a matrix

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$; i.e., H_0 must intersect the "big cell" of the Bruhat decomposition. Thus H_0 also contains:

$$\begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\det(g)}{c} \\ c & 0 \end{pmatrix}. \tag{3}$$

Let $b_0 = -\frac{\det(g)}{c}$, and $w_0 = \begin{pmatrix} 0 & b_0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\det(g)}{c} \\ c & 0 \end{pmatrix}$ If $y \in F$, we let $x = \frac{b_0}{y} c$, and we note that

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0^{-1}. \tag{4}$$

Thus we have that $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in H_0$ for all y, whence $H_0 \supset \mathrm{SL}_2(F)$.[6]

Since $H \supset H_0 \supset \mathrm{SL}_2(F)$, and H manifestly contains all scalar matrices, we see that the index of H in G_F will divide the index of $(F^\times)^2$ (i.e., the determinants of scalar matrices) in $F^\times$. This index is finite, and thus so is the index of H in G_F. Our argument is complete. $\square$

We may thus, from now on, restrict our attention to merely infinite-dimensional admissible representations. We can now state our fundamental result:

Theorem 3.2 (The Kirillov Model, c.f., Theorem 1 of [3]). *Let π be an irreducible admissible representation of G_F on an infinite dimensional vector space V. There exists one and only*

[6] As with identity (2), identities (3) and (4) are algebraically intriguing and worth internalizing. (3) states that we may conjugate an arbitrary matrix by a unipotent matrix to get an off-diagonal matrix – this is a variant of the Bruhat Decomposition. (4) states that conjugating a unipotent matrix by any off-diagonal matrix yields an opposite unipotent matrix.

one space V', consisting of (a special class of) locally constant functions from $F^\times$ to $\mathbb{C}^\times$, and one and only one representation π' of G_F on V' isomorphic to π, such that

$$\left[\left(\pi'\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\right)(\xi')\right](x) = \tau(bx)\xi'(ax) \tag{5}$$

for all $\xi' \in V'$, $a, x \in F^\times$, $b \in F$.

Furthermore, each function in V' vanishes for $|x|$ sufficiently large, and the space $\mathscr{S}(F^\times)$ of compactly-supported, locally constant functions of $F^\times$ (i.e., continuous $\xi : F^\times \to \mathbb{C}$ such that $\xi(x) = 0$ for both $|x|$ large and *$|x|$ small) is a subspace of V'. Moreover, $\mathscr{S}(F^\times)$ has finite codimension in V'.*

This theorem tells us that the representation of the mirabolic subgroup[7]:

$$\left\{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} : a \in F^\times, b \in F\right\} \subset \mathrm{GL}_2(F),$$

defined by (5) on the $\mathbb{C}$-vector space $\mathscr{S}(F^\times)$ of locally constant compactly supported $\mathbb{C}$-valued functions on $F^\times$, *uniquely* extends to a representation of $\mathrm{GL}_2(F)$ on a (potentially larger) space $V' \supset \mathscr{S}(F^\times)$ of locally constant functions $F^\times \to \mathbb{C}$. Moreover, $\mathscr{S}(F^\times)$ has finite codimension in V'. Since, as the theorem tells us, every function in V' vanishes for $|x|$ large, we see that that the only deviation of V' from $\mathscr{S}(F^\times)$ can occur with respect to the functions' behavior near $0 \in F$; we see, in fact, that $V' = \mathscr{S} \oplus \langle\xi_1\rangle \oplus \cdots \langle\xi_n\rangle$, where $\{\xi_i\}_i$ are a finite collection of functions that do not continuously extend to 0. It turns out that the only possible values for n are $n = 0, 1$, or 2 for all admissible, irreducible π (i.e., the codimension of $\mathscr{S}(F^\times)$ in V' is either 0, 1 or 2), and that the ξ_i may be chosen to be certain multiplicative characters of $O_F^\times \to \mathbb{C}$ (extended to 0 outside $\mathcal{O}_F^\times$). These will be called Jacquet characters. As in the finite field case, these are called cuspidal, special, and principle series representations, respectively.

The proof of this theorem will be covered over several subsections. To orient ourselves, we begin with some indication of how to construct the space V'.

Imagine that we *have* such a space of functions V' and such a representation π'. One benefit – perhaps the single *greatest* benefit – of having a space of functions is that for each x in the domain $F^\times$ we have a linear functional on V' given by evaluation at x. In particular, we have the canonical linear functional given by evaluation at 1, namely $L : \xi' \to \xi'(1)$.

Let us see what relation (5) tells us about the space of functions V'. We let $\xi \mapsto \xi'$ denote an isomorphism from (π, V) to (π', V'). (5) states that if $\eta = \pi\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\xi$, then

$$\eta'(x) = \tau(bx)\xi'(ax).$$

Considering the action of the unipotent and diagonal matrices in the mirabolic subgroup, and applying L (a.k.a., evaluation at 1) to both sides, we find:

[7]It is entertaining to note that that originally coined by Kazhdan and Gelfand as a portmanteau of "miraculous parabolic." If we succeed in nothing else, we hope that this document persuades our reader of the miraculousness of this subgroup.

$$\xi'(x) = L\left[\pi\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\xi\right]. \tag{6}$$

and

$$L\left[\pi\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\xi\right] = \tau(b)L(\xi). \tag{7}$$

(6) tells us that the values of the function ξ' can be completely reconstructed from applying L to the image of ξ' under the action of $\pi\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}$. (7), on the other hand, tells us something important about the kernel of L. In particular:

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\, L\left[\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi\right] dx = L(\xi)\int_{p^{-n}} dx = L(\xi)\cdot \mathrm{Vol}(\mathfrak{p}^{-n}).$$

Thus, if ξ' is in the kernel of L, we find that

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\, L\left[\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi\right] dx = 0.$$

This motivates perhaps the most important idea of the proof: we let $V_0 \subset V$ denote the subgroup of V given by the collection of vectors $\xi \in V$ such that

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi dx = 0$$

for all large n.[8] This is called the subspace of **twisted unipotent averages**. Our hope is that V_0 (or, more precisely, the corresponding subspace in V') is exactly the kernel of the linear function L. So we would hope that the space of $V_0 \subset V$ is of codimension 1, i.e., that:

$$\dim(V/V_0) = 1.$$

Proving this will constitute the bulk of our argument. Once we have accomplished this, then, picking some isomorphism of V/V_0 with $\mathbb{C}$, we may *define* our function ξ', associated to some $\xi \in V$, via:

[8] A small analytical detail should be noted here, which will apply to every subsequent integral we will consider whose integrand lies in V. Because we are integrating over a compact subgroup of F, namely $\mathfrak{p}^{-n}$, and because the stabilizer of each $\xi \in V$ is open, the integrand $\overline{\tau(x)}\pi\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\xi$ for $x \in F$ will only ever lie in a finite-dimensional subspace of V. Thus the usual definition of an integral of a function which takes values in a finite dimensional vector space suffices for our purposes. Moreover, because $\mathfrak{p}^{-n}$ is compact, we see that this integral converges (in fact, it is a finite sum).

$$\xi'(x) := \pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \xi \mod V_0. \tag{8}$$

In the meantime, let us define the space $V/V_0 := X$. (Note that a priori we do not know if $\dim(X)$ is even finite.) The formula (8) then defines, for each $\xi \in V$, a corresponding function ξ' from $F^\times$ to X, which we will take to be the definition of ξ'.

We now verify the property (5) for the space of X-valued functions defined by (8).

Lemma 3.3. *If $\eta = \pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi$, then $\eta'(x) = \tau(bx)\xi'(ax)$.*

Proof. We must show that $\pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \eta - \tau(bx)\pi \begin{pmatrix} xa & 0 \\ 0 & 1 \end{pmatrix} \xi \in V_0$; i.e., that

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)}\pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \left[\pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} - \tau(bx)\pi \begin{pmatrix} xa & 0 \\ 0 & 1 \end{pmatrix}\right] \xi dt = 0 \tag{9}$$

or

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)}\pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \left[\pi \begin{pmatrix} xa & xb \\ 0 & 1 \end{pmatrix} - \tau(bx)\pi \begin{pmatrix} xa & 0 \\ 0 & 1 \end{pmatrix}\right] \xi dt = 0 \tag{10}$$

for sufficiently large n. We would like to apply a change of variables to make the two terms in the difference cancel. We see that the first term is

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)}\pi \begin{pmatrix} xa & xb+t \\ 0 & 1 \end{pmatrix} \xi dt \tag{11}$$

while the second term is

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t-bx)}\pi \begin{pmatrix} xa & t \\ 0 & 1 \end{pmatrix} \xi dt. \tag{12}$$

Applying the change of variables $t \mapsto t - bx$ to (11) gets the two integrals to cancel identically, provided that n is so large that $bx \in \mathfrak{p}^{-n}$ (so that the domain over which we integrate is preserved). Thus, for sufficiently large n, (9) does indeed hold, as desired. □

Lemma 3.4. *Each X-valued function $\xi'(x)$ is locally constant and vanishes for $|x|$ sufficiently large.*

Proof. The idea here, like in Proposition 3.1, is to consider the intersection of the stabilizer of a vector $\xi \in V$ with the unipotent and diagonal subgroups of the mirabolic. Indeed, if

$a \in F^\times$ is sufficiently close to 1, then by smoothness of π, $\pi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \xi = \xi$. Thus by Lemma 3.3, we have that

$$\xi'(ax) = \xi'(x) \tag{13}$$

for all $x \in F^\times$. Thus ξ' is locally constant.[9]

Next we observe that there is an ideal $\mathfrak{a}$ in F such that $\pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \xi = \xi$ for all $b \in \mathfrak{a}$. Applying Lemma 3.3 again, we find that $\xi'(x) = \tau(bx)\xi'(x)$ for all x and all $b \in \mathfrak{a}$; thus $(1 - \tau(bx))\xi'(x) = 0$. If $|x|$ is so large that $\mathfrak{a}x \supsetneq \mathfrak{p}^{-d} = \ker(\tau)$, then $\tau(bx) \neq 1$ for some value of $b \in \mathfrak{a}$, whence $\xi'(x) = 0$.

□

Lemma 3.5. *The map $\xi \mapsto \xi'$ is injective.*

Proof. Let us say that $\xi' = 0$. Recall that this means that $\pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \xi \in V_0$ for all $x \neq 0$. Thus, for each $x \in F^\times$:

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)} \pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \xi dt = 0$$

for sufficiently large n (where n depends upon x). Equivalently:

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)} \pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} 1 & x^{-1}t \\ 0 & 1 \end{pmatrix} \xi dt = 0$$

for sufficiently large n (given x).[10] Thus we may pull out $\pi \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ to get:

$$\pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \left(\int_{\mathfrak{p}^{-n}} \overline{\tau(t)} \pi \begin{pmatrix} 1 & x^{-1}t \\ 0 & 1 \end{pmatrix} \xi dt \right) = 0,$$

or, since $\pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ is invertible, we may simply write

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(t)} \pi \begin{pmatrix} 1 & x^{-1}t \\ 0 & 1 \end{pmatrix} \xi dt = 0.$$

[9] In fact, (13) tells us more than than ξ' is locally constant. Given ξ' there exists a neighborhood U of 1 in $F^\times$ such that (13) holds for all $a \in U$. Restricting U if necessary, we may assume U to be of the form $1 + \mathfrak{p}^l$ for some l (depending on ξ). Then we see that ξ' is constant on the neighborhoods $x(1 + \mathfrak{p}^l)$ for all x. As $x \to 0$, we see that this implies that the oscillation of our function cannot be too violent: if $|x| = q^{-k}$, then $\xi(x)$ is constant on an open compact disk of volume q^{-k-l} containing x.

[10] Swapping the order of the diagonal and and unipotent multiplicands in the mirabolic subgroup, i.e., the matrix identity $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x^{-1}t \\ 0 & 1 \end{pmatrix}$, will be of great use to us throughout our work here. This is another algebraic identity worth internalizing.

Thus, by a change of variables $t \mapsto x^{-1}t$, we find that:

$$\int_{\mathfrak{p}^{-n}} \overline{\tau(tx)}\pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \xi dt = 0. \tag{14}$$

for n sufficiently large (depending on x, as always). We notice that this is simply the Fourier transform of $y \mapsto 1_{\mathfrak{p}^{-n}} \cdot \pi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \xi$. Thus (14) tells us that, given any $x \in F$ *except* $x = 0$, the Fourier transform of the function $y \mapsto 1_{\mathfrak{p}^{-n}} \cdot \pi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \xi$ is equal to 0 when evaluated at x, for all n sufficiently large given x. For those familiar with the theory of distributions, this should be very suggestive: we expect that $\varphi(x) := \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi$ is the inverse Fourier transform, in the sense of distributions, of something supported only at 0. This should imply that $\varphi(x)$ is constant (recall the heuristic, beloved by physicists, that the inverse Fourier transform of a Dirac delta function is a constant function).

We shall now rigorously demonstrate that $\varphi(x) := \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi$ is constant. Recall that there is a nonzero ideal $\mathfrak{a} \subset \mathfrak{p}$ such that $\pi \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \xi = \xi$ for all $u \in 1 + \mathfrak{a}$. We note that all such u are units, i.e., $|u| = 1$; moreover, the collection of all $u \in 1 + \mathfrak{a}$ form a multiplicative subgroup of $\mathcal{O}_F^\times$: closure under multiplication and identity follow easily, while inversion follows from the power series identity $(1 + a)^{-1} = 1 - a + a^2 - \cdots$ which is easily seen to converge if $a \in \mathfrak{a} \subset \mathfrak{p}$ (indeed, it is a geometric series). Define

$$\phi_n(x) := \int_{\mathfrak{p}^{-n}} \tau(xt)\varphi(t)dt.$$

We find that, for all n, t, and $u \in 1 + \mathfrak{a}$:

$$\begin{aligned}
\phi_n(xu) &= \int_{\mathfrak{p}^{-n}} \tau(xut)\varphi(t)dt \\
&= \int_{\mathfrak{p}^{-n}} \tau(xt)\varphi(u^{-1}t)\frac{dt}{|u|} \\
&= \int_{\mathfrak{p}^{-n}} \tau(xt)\pi \begin{pmatrix} 1 & u^{-1}t \\ 0 & 1 \end{pmatrix} \xi dt \\
&= \int_{\mathfrak{p}^{-n}} \tau(xt) \left(\pi \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) \xi dt \\
&= \int_{\mathfrak{p}^{-n}} \tau(xt) \left(\pi \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) \xi dt \\
&= \pi \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi_n(t).
\end{aligned}$$

By (14), we see that, given $x \in F^\times$, we will have that $\varphi_n(x) = 0$ for all n sufficiently large (to get $\overline{\tau(tx)}$ as in (14) simply replace x by $-x$). If K is a compact subset of $F^\times$, then we may find a compact set $K' \supset K$, which is a finite union of cosets mod$^*(1+\mathfrak{a})$. (Indeed, we may let $K' = K \cdot (1+\mathfrak{a})$.) Thus, picking a (finite) collection of representatives x_i for K' mod$^*(1+\mathfrak{a})$, we may find an N sufficiently large that $\varphi_n(x_i) = 0$ *for all* i if $n \geq N$. But the identity $\phi_n(xu) = \pi \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi_n(t)$ tells us that the vanishing of $\phi_n(x_i)$ is contagious: we see that $\phi_n(x) = 0$ for all $x \in (1+\mathfrak{a})x_i$; i.e., $\phi_n(x) = 0$ for all $x \in K'$ when $n \geq N$. Thus, a fortiori, $\phi_n(x) = 0$ for all $x \in K$ if $n \geq N$.

The virtue of this argument is that we now know that we can get $\phi_n(x)$ to vanish when n is sufficiently large *for all* $x \in K$ *simultaneously*, where K is an arbitrary compact subset of $F^\times$. This is better than knowing that $\phi_n(x)$ vanishes for sufficiently large n at each fixed value of x. Let us now pick a test-function $\psi \in \mathscr{S}(F)$ whose Fourier transform

$$\widehat{\psi}(x) = \int_F \overline{\tau(xt)}\psi(t)dt$$

we suppose vanishes at $x = 0$ (i.e., $\int_F \psi(t)dt = 0$). Since $\widehat{\psi}(x) \in \mathscr{S}(F)$ (a property certainly *not* enjoyed by Archimedean fields), we see that $\widehat{\psi}(x) = 0$ for all x in some open neighborhood of 0. We can therefore say that the support of $\widehat{\psi}$ is contained in some compact subset K of $F^\times$. We see that $\psi(x)$ vanishes for all x outside of $\mathfrak{p}^{-n}$ for some n since $\psi \in \mathscr{S}(F)$, and we may, increasing n if necessary, also assume that $\phi_n(t) = 0$ for all $t \in K$. Thus, we find, applying the Fourier inversion formula and Fubini's theorem, that

$$\begin{aligned}
\int_F \psi(x)\varphi(x)dx &= \int_{\mathfrak{p}^{-n}} \psi(x)\varphi(x)dx \\
&= \int_{\mathfrak{p}^{-n}} \varphi(x) \int_F \widehat{\psi}(t)\tau(tx)dtdx \\
&= \int_{\mathfrak{p}^{-n}} \varphi(x) \int_K \widehat{\psi}(t)\tau(tx)dtdx \\
&= \int_K \widehat{\psi}(t) \int_{\mathfrak{p}^{-n}} \varphi(x)\tau(tx)dxdt \\
&= \int_K \phi_n(t)\widehat{\psi}(t)dt \\
&= 0.
\end{aligned}$$

Thus the function $\varphi(x)$ is orthogonal to all functions $\psi \in \mathscr{S}(F)$ that satisfy $\int_F \psi(t)dt = 0$, i.e., all functions $\psi \in \mathscr{S}(F)$ orthogonal to the function 1. Thus $\varphi(x)$ is constant, as desired.[11]

[11] To solidify this argument, we note that, by smoothness of π, there exists some ideal $\mathfrak{b} \subset F$ such that $\varphi(x+b) = \varphi(x)$ for all $b \in \mathfrak{b}$. If, one the other hand $x, y \in F$ and $x - y \neq \mathfrak{b}$, then letting $\varphi(x) := v$ and $\varphi(y) := w$, we may integrate $\varphi(x)$ against a "Yin-Yang Function" $\psi(x) := 1_{x+\mathfrak{b}} - 1_{y+\mathfrak{b}}$ which clearly lies in

Now we observe that if $\varphi(x) := \pi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \xi$ is constant, then ξ is invariant under all unipotent operators. By Proposition 3.1 b), this implies that $\xi = 0$, and thus the $\xi' = 0 \implies \xi = 0$ as desired.

□

From now on, we will simply identify ξ with the function ξ' and drop the "prime" notation; Lemma 3.5 tells us that we lose no information doing so. Thus $\xi \in V$ will now be seen to be an X-valued function on $F^\times$, $\xi(x)$ will be understood to mean $\xi'(x)$, and $(\pi(g)\xi)(x)$ to mean $(\pi'(g)\xi')(x)$. The canonical map L defined in the preamble to Lemma 3.3 will mean $\xi \mapsto \xi(1)$ (which, we note, is a map from V to $X = V/V_0$; we do *not* yet know that L is a linear functional). By Lemma 3.3 we may write $\pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi(x) = \tau(bx)\xi(ax)$ with impunity.

Lemma 3.6. *Let us denote by $\mathscr{S}_X(F^\times)$ the space of X-valued, locally constant functions, with compact support on $F^\times$ (i.e., the collection of all locally constant $\xi : F^\times \to \mathbb{C}$ such that $\xi(x) = 0$ if $|x|$ is sufficiently small or sufficiently large). Then $\mathscr{S}_X(F^\times) \subset V$. Moreover, for all unipotent matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in F$, and all $\xi \in V$, we have that $\xi - \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \xi \in \mathscr{S}_X(F^\times)$.*

Proof. We start with the last statement. Since $\left(\xi - \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \xi\right)(x) = (\tau(bx) - 1)\xi(x)$, and $\tau(x) = 1$ on a neighborhood of 0, we see that $\xi - \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \xi$ vanishes on a neighborhood of 0, too. We already know that $\xi(x)$ vanishes for $|x|$ large by Lemma 3.4. Thus $\xi - \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \xi \in \mathscr{S}_X(F^\times)$.

Next we will show that $\mathscr{S}_X(F^\times) \subset V$. We claim that it suffices to check that, if $\mathbf{x} \in X$ and $\varphi \in \mathscr{S}(F^\times)$ is a compactly supported, locally constant, $\mathbb{C}$-valued function on $F^\times$, then the X-valued function $x \mapsto \varphi(x)\mathbf{x}$ is in V. I.e., we claim that such "1-dimensional" functions in $\mathscr{S}_X(F^\times)$ span all of $\mathscr{S}_X(F^\times)$.

Indeed: given an arbitrary function $\xi \in \mathscr{S}_X(F^\times)$, we note that the support of ξ is compact, while for all $x \in \operatorname{Supp}\xi$, identity (13) gives us that $\xi(ax) = \xi(x)$ for all $a \in 1 + \mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{p}$ is some ideal. Thus we may cover $\operatorname{Supp}\xi$ with open subsets of the form $(1 + \mathfrak{a})x$ as x ranges over all $x \in \operatorname{Supp}\xi$. By compactness, there exists a finite subcover; moreover, as demonstrated in the proof of Lemma 3.5, $1 + \mathfrak{a}$ is a multiplicative subgroup, so the multiplicative translates of $1 + \mathfrak{a}$ are disjoint. Thus, in fact, the image of ξ in X is finite (as a set), and so lies in a finite-dimensional vector-subspace $X' \subset X$. Choosing a basis $\{e_i\}_{i=1}^n$ for X', we may write $\xi(x) = \sum_{i=1}^n \varphi_i(x)e_i$, where each $\varphi_i \in \mathscr{S}(F^\times)$.[12] Thus

$\mathscr{S}(F)$ and has total integral 0 (and, since $x - y \notin \mathfrak{b}$, is nonzero). We find, applying the above orthogonality relation, that $0 = \int_F \psi(x)\varphi(x)dx = (v - w)(\mathrm{Vol}(\mathfrak{b}))$, whence $v = w$. Thus φ is constant.

[12] Our argument goes further; in fact, we may take each φ_i be the characteristic function of some multiplicative translate of $1 + \mathfrak{a}$. I.e., letting x_i be a (finite) collection of representatives $(1 + \mathfrak{a})$-multiplicative-cosets in $\operatorname{Supp}(\xi)$, we may let $\varphi_i(x) = 1_{x_i(1+\mathfrak{a})}$, and $e_i = \xi(x_i)$.

$$\mathscr{S}_X(F^\times) = \mathscr{S}(F^\times) \otimes X; \tag{15}$$

i.e., $\mathscr{S}_X(F^\times)$ is spanned by functions of the form $x \mapsto \varphi(x)v$, for $\varphi \in \mathscr{S}(F^\times)$.

For each $\mathbf{x} \in X$, we denote by $\mathscr{S}_{\mathbf{x}}(F^\times)$ the space of all functions from $F^\times$ to X of the form $\varphi \cdot \mathbf{x}$, with $\varphi \in \mathscr{S}(F^\times)$. We recall that (5) defines a representation of the mirabolic subgroup of $\mathrm{GL}_2(F)$ on $\mathscr{S}(F^\times)$. We see, moreover (applying Lemma 3.3), that the space of $\varphi \in \mathscr{S}(X)$ such that $\varphi \cdot \mathbf{x} \in \mathscr{S}_{\mathbf{x}}(F^\times) \cap V$ is preserved by the operations

$$\{x \mapsto \varphi(x)\} \mapsto \{x \mapsto \tau(bx)\varphi(ax)\}. \tag{16}$$

We claim that this representation of the mirabolic subgroup on $\mathscr{S}(F^\times)$ is irreducible.[13] Having shown this, as long as we can demonstrate that there exists some nonzero $\xi_{\mathbf{x}} \in \mathscr{S}_{\mathbf{x}}(F^\times) \cap V$ for enough $\mathbf{x}$ to span X, then the orbit of $\xi_{\mathbf{x}}$ under the mirabolic subgroup, i.e., $\pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi_{\mathbf{x}}$, spans all of $\mathscr{S}_{\mathbf{x}}(F^\times)$. This implies $\mathscr{S}_{\mathbf{x}}(F^\times) \subset V$, which we have shown implies that $\mathscr{S}_X(F^\times) \subset V$, as desired. We will postpone the proof that $\mathscr{S}_{\mathbf{x}}(F^\times) \cap V \neq 0$ until after we have proven the irreducibility claim; the two arguments are very similar.

The irreducibility of the action of the mirabolic subgroup on $\mathscr{S}(F^\times)$ given by (16) is important enough that we honor it with its own lemma:

Lemma 3.7. *The representation of the mirabolic subgroup* $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^\times, b \in F \right\} \subset \mathrm{GL}_2(F)$*, acting on* $\mathscr{S}(F^\times)$ *via*

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : \{x \mapsto \varphi(x)\} \mapsto \{x \mapsto \tau(bx)\varphi(ax)\} \tag{17}$$

is irreducible.

Proof. Let $\mathscr{H}$ be a mirabolic-subrepresentation of $\mathscr{S}(F^\times)$ under (17). Each function $\xi \in \mathscr{S}(F^\times)$ is fixed by an open subgroup of the group of units $\mathcal{O}_F^\times$, where $a \in \mathcal{O}_F^\times$ acts on ξ via $\{x \mapsto \xi(x)\} \mapsto \{x \mapsto \xi(ax)\}$.

We now define the space $\mathscr{S}(F^\times)(\chi)$, for each character χ of $\mathcal{O}_F^\times$, as the the space of all $\xi \in \mathscr{S}(F^\times)$ such that

$$\xi(ux) = \chi(u)\xi(x) \tag{18}$$

[13] Note that this is the analogue of what, in the finite field case, Piatetski-Shapiro calls V_π (see [5]). V_π is also irreducible; in fact, this representation and its irreducibility play much the same role here as they do in the classification of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

for all $u \in \mathcal{O}_F^\times$, $x \in F^\times$. We claim[14] that

$$\mathscr{S}(F^\times) = \bigoplus_{\chi} \mathscr{S}(F^\times)(\chi). \tag{19}$$

Indeed, we find that if $\xi \in \mathscr{S}(F^\times)$ is stabilized by a subgroup – say, $1 + \mathfrak{a} \subset \mathcal{O}_F^\times$ – then it is constant on the $F^\times$-cosets of $1 + \mathfrak{a}$. Since ξ is compactly supported in $F^\times$, we may break ξ up into a finite combination of indicator functions

$$\xi = \sum_{i,k} \xi(\varpi^k \cdot u_i) 1_{\varpi^k \cdot u_i \cdot (1+\mathfrak{a})} \tag{20}$$

for u_i a (finite, since $\mathcal{O}_F^\times$ is compact) collection of $(1 + \mathfrak{a})$-coset representatives in $\mathcal{O}_F^\times$ and $k \in \mathbb{Z}$. We thus find that for each fixed k, the function

$$\xi_k := \sum_{i} \xi(\varpi^k \cdot u_i) 1_{\varpi^k \cdot u_i \cdot (1+\mathfrak{a})} = \xi \cdot 1_{\{x \,:\, v_{\mathfrak{p}}(x) = k\}} \tag{21}$$

is determined by its values on the set $\{\varpi^k u_i\}_i$. Since the quotient $A := \mathcal{O}_F^\times/(1 + \mathfrak{a})$ is a finite Abelian group, we know, by elementary character theory, that all functions on A can be expressed uniquely as a finite linear combination of characters of A. Any character of A can then be lifted to a character of $\mathcal{O}_F^\times$ with kernel $1 + \mathfrak{a}$.

For each character χ of $\mathcal{O}_F^\times$, define

$$\chi_*(x) = \begin{cases} \chi(x) & \text{if } x \in \mathcal{O}_F^\times \\ 0 & \text{if } x \notin \mathcal{O}_F^\times. \end{cases} \tag{22}$$

We find that we may write

$$\xi_k(x) = \sum_{\chi} a_{k,\chi}(\xi) \chi_*(\varpi^{-k} x). \tag{23}$$

where χ ranges over all characters of $\mathcal{O}_F^\times$ with kernel containing $1 + \mathfrak{a}$ (note that such characters are in bijection with characters of A), and $a_{k,\chi}(\xi)$ are (uniquely determined) complex numbers.[15] Now we fix the character χ and sum over k:

---

[14] This claim is taken for granted in both Jacquet-Langlands and Godement. The argument is straightforward but slightly fussy. We believe that the full demonstration offers some interesting insight into the structure of $\mathscr{S}(F^\times)$. All the same, we hope that including it does not impede the flow of the proof of Lemmas 3.6 and 3.7 too drastically.

[15] We note that there is an implicit dependence between ξ and χ in our definition of $a_{k,\chi}(\xi)$: we are assuming that χ is trivial on $1 + \mathfrak{a}$, where $1 + \mathfrak{a}$ is a multiplicative group such that $\xi(ax) = \xi(x)$ for all $a \in 1 + \mathfrak{a}$. We may extend $a_{k,\chi}(\xi)$ to ξ that do not possess this property by declaring that in this case $a_{k,\chi}(\xi) = 0$. Then

$$\xi_\chi(x) := \sum_k a_{\chi,k}(\xi)\chi_*(\varpi^{-k}x). \tag{24}$$

We see that

$$\xi = \sum_\chi \xi_\chi(x) \tag{25}$$

and that $\xi_\chi(ux) = \chi(u)\xi(x)$ for all $u \in \mathcal{O}_F^\times$, $x \in F^\times$. I.e., $\xi_\chi \in \mathscr{S}(F^\times)(\chi)$. The uniqueness of this decomposition, i.e., directness of the sum in (19), follows from the linear independence of characters. Hence we have (19). Moreover, we see that in order to show $\mathscr{H} \neq 0 \implies \mathscr{H} = \mathscr{S}(F^\times)$, it suffices to prove that $\mathscr{H} \neq 0 \implies \chi_* \in \mathscr{H}$ for all $\mathcal{O}_F^\times$-characters χ.

We define $\mathscr{H}(\chi)$ to be the space $\mathscr{H} \cap \mathscr{S}(F^\times)(\chi)$. First we claim that, if $\mathscr{H} \neq 0$, there is some χ such that $\mathscr{H}(\chi) \neq 0$. Indeed, let $\xi \in \mathscr{H}$ be nonzero. We may decompose ξ into constituent ξ_χ as in (25). Pick some χ for which $\xi_\chi \neq 0$ (some such χ must exist or else $\xi = 0$ contrary to assumption). Then we may recover ξ_χ via the symmetrizing integral:

$$\xi_\chi(x) = \int_{\mathcal{O}_F^\times} \xi(xt)\overline{\chi(t)}d^*t. \tag{26}$$

This follows from decomposing $\xi(xt)$ into $\sum_\chi \sum_k a_{k,\chi}(\xi)\chi_*(\varpi^{-k}xt)$, and then applying orthogonality of characters of the finite Abelian group $A := \mathcal{O}_F^\times/(1+\mathfrak{a})$. We observe that, since $\xi \in \mathscr{H}$, so is the integral on the RHS of (26), since the integral boils down to a finite sum of a complex number times a function of the form $x \mapsto \xi(xt)$, and each of these are mirabolic-translates of ξ and so lie in $\mathscr{H}$. Thus $\xi_\chi \in \mathscr{H}(\chi)$, and by hypothesis $\xi_\chi \neq 0$.

Now we will have the first appearance of the basic motive force behind all of our arguments: Gauss sums. Let $\mathscr{H}(\chi) \neq 0$, and let $\xi \in \mathscr{H}(\chi)$ be nonzero. Then pick a distinct character $\chi' \neq \chi$. By invariance of $\mathscr{H}$ under (17), we find that the function

$$\begin{aligned}\xi'(x) &:= \int_{\mathcal{O}_F^\times} \tau(ubx)\xi(uax)\overline{\chi'(u)}d^*u \\ &= \Gamma(bx, \chi\overline{\chi'})\xi(ax)\end{aligned} \tag{27}$$

is in $\mathscr{H}$ (since the integral amount to a finite sum of mirabolic-translates of ξ) where Γ is a Gauss sum defined by

$$\Gamma(x,\lambda) = \int_{\mathcal{O}_F^\times} \tau(xu)\lambda(u)d^*u \tag{28}$$

we may write $\xi(x) = \sum_\chi \sum_k a_{k,\chi}(\xi)\chi_*(\varpi^{-k}x)$ as a sum over all characters of $\mathcal{O}_F^\times$, understanding that the a priori infinite collection of summands vanishes for all but a finite number of terms.

for all $x \in F$ and all characters λ of $\mathcal{O}_F^\times$. But if λ is a nontrivial character then the basic properties of Gauss sums show that:

$$\Gamma(x, \lambda) \neq 0 \iff v_{\mathfrak{p}}(x) = -d - \mathrm{cond}(\lambda) \tag{29}$$

where $\mathfrak{p}^{-d} = \ker(\tau)$, as always; while $\mathrm{cond}(\lambda)$ denotes the conductor of λ.

Since $\chi \neq \chi'$, we have that $\chi\overline{\chi'}$ is nontrivial, which implies that we may pick nonzero $b \in F$ such that $\Gamma(bx, \chi\overline{\chi'}) \neq 0 \iff x \in \mathcal{O}_F^\times$. Then we pick $a \in F^\times$ such that $\xi(ax) \neq 0$ if $x \in \mathcal{O}_F^\times$. We see that (27) lies in $\mathscr{H}(\chi')$ and is only supported on $\mathcal{O}_F^\times$. Thus it is a scalar multiple of χ'_*. Hence $\chi'_* \in \mathscr{H}$ for all $\chi' \neq \chi$; to see that $\chi_* \in \mathscr{H}$ as well, we simply apply the same argument but using χ' in place of χ, letting $\xi = \chi'_*$ (which we know to lie in $\mathscr{H}$ from the argument immediately prior).

We have now proved proved that all $\chi_* \in \mathscr{H}$. Thus $\mathscr{S}(F^\times)(\chi) \subset \mathscr{H}$ for all χ and so $\mathscr{H} = \mathscr{S}(F^\times)$. This completes the proof of Lemma 3.7.

□

To finish the proof of Lemma 3.6, we must show that $\mathscr{S}_{\mathbf{x}}(F^\times) \neq 0$ for enough $\mathbf{x} \in X$ to generate X.

Let $\xi \in V$ be nonzero. We know by the final statement of Lemma (3.6) (proved above) that $\xi - \pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\xi \in \mathscr{S}_X(F^\times)$, and by Proposition 3.1 b) this is nonzero for some $b \in F$. Thus we may assume without loss of generality that $\xi \in \mathscr{S}_X(F^\times) \cap V$. We may, imitating the arguments outlined in equations (20)-(26), decompose ξ into a sum of the form

$$\xi(x) = \sum_k \sum_\chi \mathbf{x}_{k,\chi}(\xi)\chi_*(\varpi^{-k}x)$$

where $\mathbf{x}_{k,\chi}$ are vectors in X. By analogy with the above, let us define $\mathscr{S}_X(F^\times)(\chi)$, for each character χ of $\mathcal{O}_F^\times$, as the the space of all $\xi \in \mathscr{S}_X(F^\times)$ such that $\xi(ux) = \chi(u)\xi(x)$ for all $u \in \mathcal{O}_F^\times$, $x \in F^\times$. As before we may break ξ up into

$$\xi = \sum_\chi \xi_\chi \tag{30}$$

where $\xi_\chi \in \mathscr{S}_X(F^\times)(\chi)$. And as before we may recover the ξ_χ from ξ from the integral

$$\xi_\chi(x) = \int_{\mathcal{O}_F^\times} \xi(xt)\overline{\chi(t)}d^*t,$$

where the integrand is understood to take values in X. Since $\xi \neq 0$ we may pick a χ such that $\xi_\chi \neq 0$. The image of ξ in X is finite since $\xi \in \mathscr{S}_X(F^\times)$, so we find that the integral is a finite sum of functions $x \mapsto \xi(xt)$, each of which are G_F-translates of ξ. Thus, there exists nonzero $\xi_\chi \in V$; i.e., $\mathscr{S}_X(F^\times)(\chi) \cap V \neq 0$. Moreover (30) shows us that collection of all $\mathscr{S}_X(F^\times)(\chi) \cap V$, as χ ranges over all characters of $\mathcal{O}_F^\times$, spans all of V.

We now show that $\mathscr{S}_{\mathbf{x}}(F^\times) \cap V \neq 0$ for all $\mathbf{x} \in X$ satisfying the following condition:

$$\text{There exists } \xi \in \mathscr{S}_X(F^\times)(\chi) \cap V \text{ for some } \chi \text{ such that } \xi(1) = \mathbf{x}. \tag{31}$$

Recall that $\xi \mapsto \xi(1)$ is the same thing as sending ξ to ξ mod V_0 (where V_0 is the subspace of twisted unipotent averages). Since the collection of $\xi \in \mathscr{S}_X(F^\times)(\chi) \cap V$ spans all of V, we see that the collection of $\mathbf{x}$ satisfying (31) spans all of $X = V/V_0$.

Say we have some nonzero $\xi \in \mathscr{S}_X(F^\times)(\chi) \cap V$. We may, replacing $\xi(x)$ by $\xi(ax)$ for some $a \in F^\times$, ensure that $\xi(1) := \mathbf{x} \neq 0$ (c.f. Lemma 3.5). Picking a distinct character χ', we repeat the above Guass sum construction, defining:

$$\begin{aligned} \xi'(x) &= \int_{\mathcal{O}_F^\times} \tau(ubx)\xi(uax)\overline{\chi'(u)}d^*u \\ &= \Gamma(bx, \chi\overline{\chi'})\xi(ax). \end{aligned} \tag{32}$$

As before, we find that $\xi' \in \mathscr{S}_X(F^\times)(\chi') \cap V$. We pick $a = 1$ and choose b so that $\Gamma(bx, \chi\overline{\chi'}) \neq 0 \iff x \in \mathcal{O}_F^\times$. Then $\xi'(x) = 0$ unless $x \in \mathcal{O}_F^\times$. On the other hand, if $x \in \mathcal{O}_F^\times$, we have $\xi'(x) = \chi'(x)\xi'(1) = \chi'(x) \cdot c\,\mathbf{x}$ for some (nonzero) complex constant c coming from the Gauss sum. I.e., $\xi' = c \cdot \chi'_*\mathbf{x}$. Therefore $\chi'_*\mathbf{x} \in \mathscr{S}_{\mathbf{x}}(F^\times) \cap V$ for all $\chi_* \neq \chi$. In particular, $\mathscr{S}_{\mathbf{x}}(F^\times) \cap V \neq 0$ for all $\mathbf{x} \in X$ satisfying (31). Thus the proof of Lemma 3.6 is complete. □

4. The Commutativity Lemma

We now come to the core component of our argument. So far, we have only really been making use of the properties of the mirabolic representation stipulated by the statement of the theorem. It is now time that we leave the familiar pastures of the mirabolic, and enter the vasty forests of GL_2.

Recall from the Bruhat decomposition that GL_2 is generated by mirabolic subgroup, scalar multiples of the identity, and the special Weyl-group element

$$w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So to extend from a representation of the mirabolic to a representation of all of GL_2, we must, as we did in the finite-field case, specify the action of scalar matrices, and specify the action of w. The former problem is not difficult: since our representation is irreducible, and scalar matrices are central, we may apply Schur's theorem, and note that there exists a "central character" $\omega_\pi : F^\times \to \mathbb{C}$ such that

$$\omega_\pi(t)1 = \pi \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}. \tag{33}$$

Note that ω_π is locally constant and equals 1 in a neighborhood of $t = 1$.

So we now turn our attention to the really interesting business: the action of w. It suffices to specify this action, of course, on our favorite basis[16] for the space of locally constant X-valued functions on $F^\times$: the set of (appropriately scaled) characters χ times an element of $\mathbf{x} \in X$. Indeed, let us define the function $\chi_{t,\mathbf{x}} : F^\times \to X$ via

$$\chi_{t,\mathbf{x}}(x) : \begin{cases} \chi(t^{-1}x)\mathbf{x} & \text{if } x \in \mathcal{O}_F^\times \\ 0 & \text{if } x \notin t\mathcal{O}_F^\times \end{cases} \tag{34}$$

for all $t \in F^\times$ and $\mathbf{x} \in X$, where χ is a (continuous) character[17] of $\mathcal{O}_F^\times$. If we drop the t subscript, and just write $\chi_{\mathbf{x}}$, we take this to mean $\chi_{1,\mathbf{x}}$. Note that

$$\chi_{t,\mathbf{x}}(x) = \chi_{\mathbf{x}}(t^{-1}x). \tag{35}$$

As shown above, these functions span all of $\mathscr{S}_X(F^\times)$. The Fourier inversion formula on $F^\times$ tells us[18] that, for $\xi \in \mathscr{S}_X(F^\times)$, we have

$$\xi = \sum_{t\mathcal{O}_F^\times} \sum_{\chi} \chi_{t,\mathbf{x}} \tag{36}$$

where

$$\mathbf{x} := \mathbf{x}(t,\chi) := \int_{\mathcal{O}_F^\times} \xi(tu)\overline{\chi(u)} d^*u \tag{37}$$

where we assume that the measure d^*u is the unique Haar measure on $F^\times$ that assigns a volume of 1 to $\mathcal{O}_F^\times$.

Now, we define a linear endomorphism $J_\pi(t,\chi) : X \to X$ for each $t \in F^\times$ character χ of $\mathcal{O}_F^\times$, via[19]:

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[16] The perceptive reader might object to our use of the term "basis" here. This is not unwarranted. The functions $\chi_{t,\mathbf{x}}$ are not even linearly independent – clearly $\chi_{t,\mathbf{x}}$ and $\chi_{t,\lambda\mathbf{x}}$ are scalar multiples. We can of course restrict to those $\mathbf{x}_i$ forming a basis for X, and restrict to those t of the form ϖ^k for integers k. In this case we do indeed get a basis, but only for the space $\mathscr{S}_X(F^\times)$; it is not quite a basis for V (or for the much larger space of all locally constant X-valued functions on $F^\times$), since this will contain certain infinite sums of the functions $\chi_{\varpi^k,\mathbf{x}_i}$.

[17] It is worth noting that such characters of $\mathcal{O}_F^\times$ are always unitary. Indeed, the image of χ must be a compact multiplicative subgroup of $\mathbb{C}^\times$, and the set of norms $|\chi(x)|$ must be a compact multiplicative subgroup of $\mathbb{R}^+$. Thus $|\chi(x)| = 1$ and χ is unitary.

[18] We may of course skirt the use of Fourier inversion here and apply the work done above to decompose $\mathscr{S}_X(F^\times)$ into a basis of functions $\chi_{\varpi^k,\mathbf{x}_i}$. But it is worth noting that such a decomposition is really arising from the general Fourier inversion formula on $F^\times$.

[19] The notation J here is taken from Godement. It is meant to insinuate Bessel functions. These operators J will play an analogous role to the function j found in [5] in his explicit construction of absolutely cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$. In [5], j was a finite-field analogue of the Bessel function, defined by a sum very

$$J_\pi(t,\chi) : \mathbf{x} \mapsto (\pi(w)\chi_{t,\mathbf{x}})(1) = L[\pi(w)\chi_{t,\mathbf{x}}]. \tag{38}$$

We note that this operator is linear since it is the composition of linear operators; note that the assignment $\mathbf{x} \mapsto \chi_{t,\mathbf{x}}$ is a linear map from X to $\mathscr{S}_X(F^\times)$.

We now compute what the defining relations on GL_2 tell us about these operators J. Firstly, by (35), we have:

$$\begin{aligned} J_\pi(t,\chi)\mathbf{x} &= L\left[\pi(w)\pi\begin{pmatrix}1/t & 0\\ 0 & 1\end{pmatrix}\chi_\mathbf{x}\right] \\ &= L\left[\pi\begin{pmatrix}1 & 0\\ 0 & 1/t\end{pmatrix}\pi(w)\chi_\mathbf{x}\right] \\ &= L\left[\pi\begin{pmatrix}1/t & 0\\ 0 & 1/t\end{pmatrix}\pi\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\pi(w)\chi_\mathbf{x}\right] \\ &= \omega_\pi(t^{-1})\cdot(\pi(w)\chi_\mathbf{x})(t). \end{aligned} \tag{39}$$

Since $\chi_\mathbf{x} \in \mathscr{S}_X(F^\times) \subset V$ by Lemma 3.6, we see that the function $t \mapsto J_\pi(t,\chi)\mathbf{x}$, which equals $\pi(w)\pi\begin{pmatrix}1/t & 0\\ 0 & 1\end{pmatrix}\chi_\mathbf{x}$, also lies in V. (However, it is worth noting that $J_\pi(t,\chi)\mathbf{x}$ may not, and generally will not, be in $\mathscr{S}_X(F^\times)$.) Applying Lemma 3.4 we see that $t \mapsto J_\pi(t,\chi)\mathbf{x}$ is locally constant, and vanishes for $|t|$ sufficiently large. (Alas, we will not, in general, have that $t \mapsto J_\pi(t,\chi)\mathbf{x}$ vanishes for $|t|$ sufficiently small.)

We also find that $t \mapsto J_\pi(t,\chi)\mathbf{x}$ lies in[20] $V(\overline{\chi})$:

$$J_\pi(tu,\chi) : \mathbf{x} \mapsto (\pi(w)\chi_{tu,\mathbf{x}})(1) = \left(\pi(w)\overline{\chi(u)}\chi_{t,\mathbf{x}}\right)(1) = \left[\overline{\chi(u)}J_\pi(t,\chi)\right](\mathbf{x})$$

whence

$$J_\pi(tu,\chi) = \overline{\chi(u)}J_\pi(t,\chi). \tag{40}$$

Applying $\pi(w)$ to the Fourier inversion formula (36), and evaluating at 1, we find that if $\xi \in \mathscr{S}_X(F^\times)$, then

similar to the contour-integral representaiton of Bessel functions of the first kind; moreover, convolving with j gave us (roughly speaking) the action of the element $w \in \mathrm{GL}_2(\mathbb{F}_1)$. Unfortunately, unlike in the finite field case we cannot immediately write down a single function $J(t)$ which gives us an integral kernel for the action of $\pi(w)$; we can only write J "character-locally" as $J(t,\chi)$.

[20] We cannot yet conclude, however, that it lies in $V_\mathbf{x}(\overline{\chi})$ since the operator $J_\pi(t,\chi)$ might not preserve the subspace of X generated by $\mathbf{x}$. (But recall that in the end, X will turn out to be one-dimensional, so all nonzero vectors will be scalar multiples of one another.)

$$(\pi(w)\xi)(1) = \sum_{t\mathcal{O}_F^\times} \sum_{\chi} J_\pi(t,\chi)\,[\mathbf{x}(t,\chi)]$$
$$= \sum_{t\mathcal{O}_F^\times} \sum_{\chi} J_\pi(t,\chi) \left[\int_{\mathcal{O}_F^\times} \xi(tu)\overline{\chi(u)} d^*u \right]$$

which, subsuming the $\sum_{t\mathcal{O}_F^\times}$ into the integral, gives us the convolution formula:

(41)
$$(\pi(w)\xi)(1) = \sum_{\chi} \int_{F^\times} J_\pi(y,\chi)\xi(y)d^*y.$$

We would of course like to be able to write $J(y) = \sum_\chi J_\pi(y,\chi)$ and then simplify the above to

(42)
$$(\pi(w)\xi)(1) = \int_{F^\times} J_\pi(y)\xi(y)d^*y,$$

but we do not yet have the absolute convergence of the sum (41). To obtain the formula for $(\pi(w)\xi)(x)$ for an arbitrary $x \in F^\times$, we plug $\pi \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \xi$ in place of ξ into (41). The left hand side becomes

$$\begin{aligned}
\pi(w)\pi \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \xi &= \pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pi \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \xi \\
&= \pi \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi \\
&= \pi \begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix} \pi \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi \\
&= \omega_\pi(x^{-1})(\pi(w)\xi)(x),
\end{aligned}$$

while the right hand side becomes

$$\begin{aligned}
\sum_\chi \int_{F^\times} J_\pi(y,\chi) \left(\pi \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \xi \right)(y) d^*y &= \sum_\chi \int_{F^\times} J_\pi(y,\chi)\xi(x^{-1}y)d^*y \\
&= \sum_\chi \int_{F^\times} J_\pi(xy,\chi)\xi(y)d^*y
\end{aligned}$$

by change of variables. Thus we obtain the convlution formula:

$$(\pi(w)\xi)(x) = \omega_\pi(x) \sum_\chi \int_{F^\times} J_\pi(xy,\chi)\xi(y)d^*y. \tag{43}$$

We note that for $\xi \in \mathscr{S}_X(F^\times)$ the integral is only supported on a compact subdomain of $F^\times$, and so manifestly converges. (We might recall that $J_\pi(xy,\chi)$ vanishes for sufficiently large $|y|$; but, as discussed before, we do not have convergence of $J_\pi(xy,\chi)$ as $y \to 0$. Thus we must require the test-function ξ to lie in $\mathscr{S}_X(F^\times)$ to have convergence.) Moreover, the sum vanishes for all but finitely many characters χ.

We now come to the central lemma of the theorem.

Lemma 4.1. *All operators $J_\pi(x,\chi)$ commute.*

Proof. This lemma results from following tracing through the consequences of the final and most complicated relation on w. Let us define $u(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for all $t \in F$, and define the diagonal matrix $h(t) := \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ for all $t \in F^\times$. We have the fundamental relation[21]

$$wu(t)w^{-1} = u(-1/t)wh(t)u(-1/t) \tag{44}$$

which follows from applying the Bruhat decomposition to the left hand side.

We now compute the effect of each side of (44) on an arbitrary function $\xi \in \mathscr{S}_X(F^\times)$. We begin with the right-hand side, as we will encounter a small-but-easily-remediable technicality on the left hand side.

First, we apply $u(-1/t)$, which sends $x \mapsto \xi(x)$ to $x \mapsto \tau(-x/t)\xi(x)$. Applying $h(t) = \begin{pmatrix} 1/t & 0 \\ 0 & 1/t \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ to this function, we replace x by t^2x throughout and then scale the result by $\omega_\pi(t)$. We obtain $x \mapsto \omega_\pi(1/t)\tau(-tx)\xi(t^2x)$. Now we apply $\pi(w)$ to this, and invoking formula (43), we find that we get

$$x \mapsto \omega_\pi(x) \sum_\chi \int_{F^\times} J_\pi(xy,\chi)\omega_\pi(1/t)\tau(-ty)\xi(t^2y)d^*y.$$

Finally, applying $u(-1/t)$ to this function; i.e., multiplying by $\tau(-x/t)$, we get:

$$\begin{aligned} x &\mapsto \tau(-x/t)\,\omega_\pi(x) \sum_\chi \int_{F^\times} J_\pi(xy,\chi)\omega_\pi(1/t)\tau(-ty)\xi(t^2y)d^*y \\ &= \omega_\pi(x/t) \sum_\chi \int_{F^\times} J_\pi(xy/t^2,\chi)\tau[-t^{-1}(x+y)]\xi(y)d^*y, \end{aligned} \tag{45}$$

[21] Note that both sides are equal to the "opposite" unipotent matrix $\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$.

with the last equality coming from the change of variables $y \mapsto t^2 y$.

Now we compute the effect π of the left hand side of (44) on ξ. There is a slight issue we will encounter, however: we know from (43) what $\pi(w^{-1}) = -\pi(w)$ will do to ξ, and then from the specified mirabolic action what $u(t)$ will do to that. But we cannot immediately determine what $\pi(w)$ will do to *that*: once we have applied w to ξ once, the resulting function *is no longer guaranteed to be in* $\mathscr{S}_X(F^\times)$. Thus we cannot apply formula (43) to $\pi[u(t)w^{-1}]\xi$, and so we are in the dark with respect to how to proceed.

To get around this obstacle, we will perform an ingenious and useful trick: we note that

$$\begin{aligned} \pi[wu(t)w^{-1}]\xi &= \pi(w)[\pi(u(t))\pi(w^{-1})\xi - \pi(w^{-1})\xi] + \xi \\ &= \omega_\pi(-1)\pi(w)[\pi(u(t))\pi(w)\xi - \pi(w)\xi] + \xi. \end{aligned} \tag{46}$$

The virtue of (46) is that, by the final part Lemma (3.6), the expression in the brackets is in $\mathscr{S}_X(F^\times)$. Thus we are perfectly entitled to apply (43) to determine the action of $\pi(w)$ on the function in brackets.

Proceeding as we did before, we find that the expression in brackets is given by

$$x \mapsto \omega_\pi(x) \sum_\chi \int_{F^\times} J_\pi(xy, \chi)\xi(y)[\tau(tx) - 1] d^*y.$$

Applying $\omega_\pi(-1)\pi(w)$ to this function yields:

$$\begin{aligned} x \mapsto & \omega_\pi(-1)\omega_\pi(x) \sum_{\chi'} \int_{F^\times} J_\pi(xz, \chi') \left(\omega_\pi(z) \sum_\chi \int_{F^\times} J_\pi(zy, \chi)\xi(y)[\tau(tz) - 1] d^*y \right) d^*z \\ &= \omega_\pi(-x) \sum_{\chi', \chi} \iint_{F^\times \times F^\times} J_\pi(xz, \chi') J_\pi(zy, \chi)\xi(y)[\tau(tz) - 1]\omega_\pi(z) d^*y\, d^*z \end{aligned}$$

where swapping the order of the integrals and sums is legitimate since the internal sum is, in fact, finite.[22] Adding the remaining ξ term, we find that

$$\begin{aligned} &(\pi[wu(t)w^{-1}]\xi)(x) = \\ &\qquad \xi(x) + \omega_\pi(-x) \sum_{\chi', \chi} \iint J_\pi(xz, \chi') J_\pi(zy, \chi)\xi(y)[\tau(tz) - 1]\omega_\pi(z) d^*y\, d^*z. \end{aligned} \tag{47}$$

Let us call this function $\eta_t(x)$; note that (45) gives us "the other" formula for $\eta_t(x)$.

Let $t_1, t_2 \in F^\times$. Computing $\eta_{t_1}(x) - \eta_{t_2}(x)$ using (45) and (47), we find (after cancelling $\omega_\pi(-x)$ from each side) that

[22] This is because the difference $\tau(tz) - 1$ vanishes at (and in some neighborhood of) $z = 0$. Thus the integral will converge for each fixed χ, χ'; moreover, the integral will vanish if the conductor of the product $\chi'\chi$ becomes sufficiently large. Thus the sum is in fact finite for each fixed x.

$$\begin{aligned} &\omega_\pi(-t_1^{-1}) \sum_\chi \int_{F^\times} J_\pi(xy/t_1^2, \chi)\tau[-t_1^{-1}(x+y)]\xi(y)d^*y \\ &- \omega_\pi(-t_2^{-1}) \sum_\chi \int_{F^\times} J_\pi(xy/t_2^2, \chi)\tau[-t_2^{-1}(x+y)]\xi(y)d^*y \\ &= \sum_{\chi',\chi} \iint J_\pi(xz, \chi')J_\pi(zy, \chi)\xi(y)[\tau(t_1 z) - \tau(t_2 z)]\omega_\pi(z)d^*y\, d^*z. \end{aligned} \tag{48}$$

This formula may at first seem to be an unseemly mess; but it is, upon closer inspection, one of the most beautiful and important identities we have seen thus far. We see that the formula gives the equality of the action of two integral kernels on a function ξ. The left hand side, for instance, has kernel

$$\begin{aligned} K(x, y) := &\ \omega_\pi(-t_1^{-1})J_\pi(xy/t_1^2, \chi)\tau[-t_1^{-1}(x+y)] \\ &- \omega_\pi(-t_2^{-1})J_\pi(xy/t_2^2, \chi)\tau[-t_2^{-1}(x+y)] \end{aligned}$$

which takes values in $\mathrm{End}(X)$. And now we make what appears to be a totally innocent observation, but in fact is the single most important insight in this entire proof: *this kernel is symmetric in x and y.*

An immediate consequence of this observation is that the (equal) kernel on the right hand side of (48), i.e.,

$$K(x, y) = \sum_{\chi',\chi} \int J_\pi(xz, \chi')J_\pi(zy, \chi)[\tau(t_1 z) - \tau(t_2 z)]\omega_\pi(z)\, d^*z, \tag{49}$$

is *also* symmetric in x an y.[23] But this is not a priori obvious like was on the left hand side – it tells us something new! Indeed, swapping x and y yields

$$K(y, x) = \sum_{\chi,\chi'} \int J_\pi(zy, \chi)J_\pi(xz, \chi')[\tau(t_1 z) - \tau(t_2 z)]\omega_\pi(z)\, d^*z, \tag{50}$$

which *almost* looks like it is the same expression – except that *the two J operators are interchanged*! Since the J's are not yet known to commute, this is indeed a most astonishing revelation.

The remainder of this section is devoted to unwinding the equality of the integrals (49) and (50), in order to obtain the equality of the operators $J_\pi(xz, \chi')J_\pi(zy, \chi)$ and $J_\pi(zy, \chi)J_\pi(xz, \chi')$. We have that:

[23] We note, as in footnote 22, that the difference $\tau(t_1 z) - \tau(t_2 z)$ vanishes at (and in some neighborhood of) $z = 0$. Thus the integral will converge for each pair of characters χ, χ'; moreover, the integral will vanish if the conductor of $\chi\chi'$ becomes large. Hence the sum is in fact finite for each fixed x, y, t_1 and t_2.

$$\sum_{\chi',\chi} \int J_\pi(xz,\chi')J_\pi(zy,\chi)[\tau(t_1z)-\tau(t_2z)]\omega_\pi(z)\,d^*z$$
$$= \sum_{\chi,\chi'} \int J_\pi(zy,\chi)J_\pi(xz,\chi')[\tau(t_1z)-\tau(t_2z)]\omega_\pi(z)\,d^*z, \tag{51}$$

which is actually an equality of finite sums. Applying the transform $x \mapsto xu$ or $y \mapsto yu$ for $u \in \mathcal{O}_F^\times$ (which multiplies each summand by $\overline{\chi'(u)}$ and $\overline{\chi(u)}$ respectively), and recalling the linear independence of characters, we see that each individual integral must agree:

$$\int J_\pi(xz,\chi')J_\pi(zy,\chi)[\tau(t_1z)-\tau(t_2z)]\omega_\pi(z)\,d^*z$$
$$= \int J_\pi(zy,\chi)J_\pi(xz,\chi')[\tau(t_1z)-\tau(t_2z)]\omega_\pi(z)\,d^*z, \tag{52}$$

where χ and χ' are fixed characters of $\mathcal{O}_F^\times$. Let

$$\varphi(z) := \omega_\pi(z)[J_\pi(xz,\chi')J_\pi(yz,\chi) - J_\pi(yz,\chi)J_\pi(xz,\chi')], \tag{53}$$

which takes values in $\mathrm{End}(X)$. We see that

$$\int_{F^\times} \varphi(z)[\tau(t_1z)-\tau(t_2z)]d^*z = 0 \tag{54}$$

for all t_1 and t_2 in $F^\times$. Note that we may extend the integrand to all of F by letting $\varphi(z)[\tau(t_1z)-\tau(t_2z)] = 0$ at $z = 0$; we know, after all, that $\tau(t_1z)-\tau(t_2z)$ vanishes on a neighborhood of $z = 0$. Since, for each $\mathbf{x} \in X$, the X-valued function on $F^\times$ given by $z \mapsto \varphi(z)\mathbf{x}$ is locally constant and compactly supported, we see that (54) tells us that $z \mapsto \varphi(z)\mathbf{x}$ has an inverse Fourier transform that is constant. Thus $z \mapsto \varphi(z)\mathbf{x} = 0$ for all $z \in F^\times$, and so $J_\pi(xz,\chi')J_\pi(yz,\chi) = J_\pi(yz,\chi)J_\pi(xz,\chi')$ for all x, y, z, χ, χ'.

□

Now we may reap the rewards for our labor.

Lemma 4.2. *The space X is 1 dimensional.*

Proof. First we observe that

$$V = \mathscr{S}_X(F^\times) + \pi(w)\mathscr{S}_X(F^\times) \tag{55}$$

Indeed, let us denote the space on right hand side by V' (a priori a subspace of V). We check that V' is G_F-invariant. By the Bruhat decomposition, we know that it suffices to check that V' is preserved by the scalar operators λI, the mirabolic operators $\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, and the operator $\pi(w)$. Since the scalar matrices λI act by scalars $\omega_\pi(\lambda)$, we see that V' is clearly stable under these operators. We also see that $\pi(w)$ preserves V': since $\pi(w)^2 = \omega_\pi(-1)I$, we see that applying $\pi(w)$ to $\mathscr{S}_X(F^\times) + \pi(w)\mathscr{S}_X(F^\times)$ merely swaps the two addends. For the mirabolic, we break up into cases $\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\pi(u(b)) = \pi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Since

$$\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\pi(w) = \omega_\pi(a)\pi(w)\pi\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

and the operators $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ preserve $\mathscr{S}_X(F^\times)$, we see that $\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}V' = V'$. Finally we check invariance under the unipotents. If $\xi \in \mathscr{S}_X(F^\times)$, then $\pi(u(b))\pi(w)\xi - \pi(w)\xi \in \mathscr{S}_X(F^\times)$ by the last part of Lemma 3.6. Thus $\pi(u(b))\pi(w)\xi \in V'$, and we see that V' is stable under all operators in G_F. Thus $V' = V$.

Next, we assert that any linear operator A on X that commutes with every $J_\pi(x, \chi)$ is a scalar. Indeed, say A were such an operator. Then we define an operator T_A, on the space of locally constant functions from $F^\times$ to X, by $(T_A\xi)(x) = A(\xi(x))$.[24] We first claim that T_A commutes with the action of the mirabolic. Indeed:

$$\begin{aligned}
\left(T_A\,\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi\right)(x) &= A\left(\left(\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi\right)(x)\right) \\
&= A\left(\tau(ax)\xi(bx)\right) \\
&= \tau(ax)A(\xi(bx)) \\
&= \tau(ax)(T_A(\xi))(bx) \\
&= \left(\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}T_A\xi\right)(x).
\end{aligned}$$

We note that $\mathscr{S}_X(F^\times)$ is certainly preserved by the action of the operator T_A; thus, by Lemma 3.7 we see that T_A must act via a scalar on the space $\mathscr{S}_{\mathbf{x}}(F^\times)$ for each $\mathbf{x} \in X$. Thus T_A must act as a single scalar, say λ, on all of $\mathscr{S}_X(F^\times)$.[25]

Now, from above, we know that we may write an arbitrary $\xi \in V$ as $\xi = \xi' + \pi(w)\xi''$, where $\xi', \xi'' \in \mathscr{S}_X(F^\times)$. We find:

[24] Note that we do not know whether $T_A\xi \in V$ for all $\xi \in V$.

[25] Here we use the linear algebra fact that if a linear operator scales each vector in a vector space then it is itself a scalar operator.

$$\begin{aligned}(T_A\xi)(x) &= (T_A(\xi' + \pi(w)\xi''))(x) \\ &= \lambda\xi' + (T_A\pi(w)\xi'')(x) \\ &= \lambda\xi' + A\left(\omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)\xi''(y)d^*y\right).\end{aligned}$$

Recalling that the sum is, in fact, finite, and using the fact that A commutes with all the $J_\pi(xy,\chi)$, we obtain

$$\begin{aligned}\lambda\xi'(x) + \omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)A(\xi''(y))d^*y &= \lambda\xi'(x) + \omega_\pi(x)\sum_\chi \int_{F^\times} J_\pi(xy,\chi)\lambda\xi''(y)d^*y \\ &= \lambda\xi.\end{aligned}$$

Thus T_A acts as a scalar (by λ) on all of V. Hence, if $\mathbf{x} \in X$, and $\xi \in V$ is a vector which equals $\mathbf{x}$ mod twisted unipotent averages (i.e., $\xi(1) = \mathbf{x}$), then we see that $A\mathbf{x} = A(\xi(1)) = (T_A\xi)(1) = (\lambda\xi)(1) = \lambda\mathbf{x}$. Thus A is a scalar on X.

Finally, we apply Lemma 4.1, and see that each of the $J_\pi(x,\chi)$ are scalars, since they all commute with one another. Thus every operator on X commutes with all the $J_\pi(x,\chi)$, whence every operator on X is a scalar. Thus X is 1 dimensional.

□

5. Finite Codimensionality of $\mathscr{S}(F^\times)$ and Uniqueness of the Kirillov Model

We have just proved that X is 1-dimensional, and thus we may identify (up to an ambiguous scalar multiple) each erstwhile X-valued function ξ on $F^\times$ with a $\mathbb{C}$-valued function on $F^\times$. Let us collect our results so far. For every irreducible, admissible, infinite-dimensional representaiton π, we have shown that we may view π as acting on V, a space of $\mathbb{C}$-valued functions on $F^\times$, which, by Lemma 3.3, has the specified mirabolic action (5). By Lemma 3.4, each such function is locally constant, and vanishes outside of some compact subset of F; by Lemma 3.6 we know that $\mathscr{S}(F^\times) \subset V$. We now need to demonstrate that $\mathscr{S}(F^\times)$ is of finite codimension in V.

To simplify notation, let $V_* := \mathscr{S}(F^\times)$. We know by (55) that $V = V_* + \pi(w)V_*$. Quotienting by V_*, we find that we need to show that

$$\dim[V_*/(\pi(w)V_* \cap V_*)] < \infty. \tag{56}$$

We recall from (19) our decomposition $V_* = \bigoplus_\chi V_*(\chi)$ for each character χ of $\mathcal{O}_F^\times$. We define a canonical map

$$V_*/\left(\pi(w)V_*\cap V_*\right)\to\bigoplus_\chi V_*(\chi)/\left[\pi(w)V_*\cap V_*(\chi)\right],$$

given by projecting $\xi\in V_*$ onto its $V_*(\chi)$ components. We must check that this map is well-defined. If $\xi\in\pi(w)V_*\cap V_*$, we may, by (25), decompose ξ into $\sum_\chi \xi_\chi$ with each $\xi_\chi\in V_*(\chi)$. We now show that each $\xi_\chi\in V_*(\chi)/[\pi(w)V_*\cap V_*(\chi)]$. Indeed, since $\xi\in\pi(w)V_*\cap V_*$, we see that $V_*\ni\eta:=\pi(w)^{-1}\xi$. Thus η can be decomposed into χ-isotypic components: $\eta=\sum_\chi\eta_\chi$ with each $\eta_\chi\in V_*(\chi)$. Applying $\pi(w)$ to this sum, we find that $\xi=\sum_\chi\pi(w)\eta_\chi$. Now we note that $\pi(w)\eta_\chi\in V(\omega_\pi\chi^{-1})$; indeed: $\left(\pi(w)\eta_\chi\right)(ux)=\left(\begin{pmatrix}u&0\\0&1\end{pmatrix}\pi(w)\eta_\chi\right)(x)=\omega_\pi(u)\left(\pi(w)\pi\begin{pmatrix}u^{-1}&0\\0&1\end{pmatrix}\eta_\chi\right)(x)=(\omega_\pi\chi)^{-1}(u)(\pi(w)\eta_\chi)(x)$. Thus $\pi(w)$ sends $V(\omega_\pi\chi^{-1})$ to $V(\chi)$ (in fact, it interchanges them). Hence $\pi(w)\eta=\xi$ becomes $\sum_\chi\pi(w)\eta=\sum_\chi\xi_{\omega_\pi\chi^{-1}}$; by the uniqueness of the character-decomposition for ξ, we find that $\pi(w)\eta_\chi=\xi_{\omega_\pi\chi^{-1}}$. Thus $\xi_\chi\in\pi(w)V_*\cap V_*(\chi)$ for each χ, and the above map is well-defined.

There is also a map in the other direction induced by the inclusions $V_*(\chi)\hookrightarrow V_*$ for each χ. The resulting homomorphism clearly well-defined, and inverts the map discussed in the previous paragraph. Hence

$$V_*/\left(\pi(w)V_*\cap V_*\right)=\bigoplus_\chi V_*(\chi)/\left[\pi(w)V_*\cap V_*(\chi)\right].\tag{57}$$

Thus, to prove finite dimensionality of the quotient on the LHS of (57), we must demonstrate two things: 1) that $V_*(\chi)/\left[\pi(w)V_*\cap V_*(\chi)\right]$ is finite dimensional for all χ, and 2) that $V_*(\chi)/\left[\pi(w)V_*\cap V_*(\chi)\right]=0$ for all but a finite number of characters χ. This in itself is a rather wonderful fact – a priori it seems stronger than the finite codimensionality of V_*. (In reality, however, the above does show that they are equivalent.) It also draws our attention to an important finite collection of characters of $\mathcal{O}_F^\times$ that is canonically associated to a given infinite dimensional admissible representation of $\mathrm{GL}_2(F)$ – namely, those χ for which $V_*(\chi)/\left[\pi(w)V_*\cap V_*(\chi)\right]\neq 0$.

At any rate, to prove these claims we shall first prove the following:

Lemma 5.1. *For each character χ of $\mathcal{O}_F^\times$, we have $\pi(w)V_*\cap V_*(\chi)\neq 0$.*

Proof. If $\pi(w)V_*=V_*$, then clearly we are already done: $\pi(w)V_*\cap V_*(\chi)=V_*(\chi)\neq 0$. So let us suppose that $\pi(w)V_*\neq V_*$, as shown in the proof of Lemma 3.7.

In this case, we will explicitly construct a nonzero $\xi\in\pi(w)V_*\cap V_*$, As always, Gauss sums will come to our rescue. Let $\xi\in V_*$ be nonzero. We cook up the following ingenious expression:

$$\pi[u(t)w^{-1}]\xi-\pi(w^{-1})\xi-\pi[h(t)u(-1/t)]\xi.\tag{58}$$

We claim that this lies in $\pi(w)V_* \cap V_*$. The difference $\pi[u(t)w^{-1}]\xi - \pi(w^{-1})\xi \in V_*$, by Lemma 3.6, and $\pi[h(t)u(-1/t)]\xi \in V_*$ by Lemma 3.3. Thus (58) certainly lies in V_*. To show that (58) also lies in $\pi(w)V_* = \pi(w^{-1})V_*$, we applying $\pi(w)$ to both sides, yielding:

$$\pi[wu(t)w^{-1}]\xi - \xi - \pi[wh(t)u(-1/t)]\xi \tag{59}$$

and so by the fundamental relation (44) we obtain:

$$\pi[u(-1/t)wh(t)u(-1/t)]\xi - \xi - \pi[wh(t)u(-1/t)]\xi. \tag{60}$$

Now we find that the difference in the first and third term lies in V_* by Lemma 3.6, while $\xi \in V_*$ by assumption. Thus (58) lies in $\pi(w)V_* \cap V_*$. Our objective now is to specialize t and ξ so that the usual symmetrization into $V_*(\chi)$ is nonzero.

Evaluating (58) at $x \in F^\times$ yields:

$$\omega_\pi(-1)[\tau(tx) - 1](\pi(w)\xi)(x) - \omega_\pi(t^{-1})\tau(-tx)\xi(t^2x). \tag{61}$$

Replacing x with xu in the above function is tantamount to applying the operator $\pi\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} = \omega_\pi(u)h(u)$, and this operator preserves $\pi(w)V_* \cap V_*$. Thus the function

$$\begin{aligned} x \mapsto \omega_\pi(-1) &\int [\tau(txu) - 1]\pi(w)\xi(xu)\overline{\chi}(u)d^*u \\ - \omega_\pi(t^{-1}) &\int \tau(-txu)\xi(t^2xu)\overline{\chi}(u)d^*u \end{aligned} \tag{62}$$

still lies in $\pi(w)V_* \cap V_*$. Now we let

$$\xi(x) = \chi'_*(x) := \begin{cases} \chi'(x) & \text{if } x \in \mathcal{O}_F^\times \\ 0 & \text{if } x \notin \mathcal{O}_F^\times, \end{cases}$$

for a character χ' of $\mathcal{O}_F^\times$ not equal to χ. We recall that

$$(\pi(w)\xi)(x) = \omega_\pi(x)J_\pi(x, \chi'),$$

and that $\pi(w)\xi \in V(\omega_\pi\chi'^{-1})$.[26] Thus $V_* \cap \pi(w)V_*$ contains the function

[26]Note, however, that $\pi(w)\xi$ is *not* necessarily in $V_*(\omega_\pi\chi'^{-1})$; indeed, we do not necessarily have that $\pi(w)\xi \in V_*$.

$$x \mapsto \omega_\pi(-1)J_\pi(x,\chi')\int[\tau(txu)-1]\omega_\pi(u)\overline{\chi'\chi}(u)d^*u$$

$$- \omega_\pi(t^{-1})\chi'_*(t^2x)\int \tau(-txu)\chi'\overline{\chi}(u)d^*u, \tag{63}$$

which equals

$$\big(\omega_\pi(x)J_\pi(x,\chi')[\Gamma(tx,\omega_\pi\overline{\chi'\chi})-\delta(\omega_\pi\overline{\chi'\chi})]\big)-\omega_\pi(t^{-1})\chi'_*(t^2x)\Gamma(-tx,\overline{\chi'\chi}) \tag{64}$$

where Γ is a Gauss sum as in (28), and $\delta(\chi)=0$ unless χ is trivial in which case $\delta(\chi)=1$. We are done if we may esnure that (64) does not vanish for some well-chosen χ', x and t. Luckily, we have very specific control over the vanishing and nonvanishing of Gauss sums, so this is merely a matter of carefulness.

We may declare victory if we find χ', x and t such that

$$\begin{gathered} J_\pi(x,\chi')\neq 0 \\ \Gamma(tx,\omega\overline{\chi'\chi})-\delta(\omega_\pi\overline{\chi\chi'})\neq 0 \\ 2v(t)+v(x)\neq 0, \end{gathered} \tag{65}$$

as this will force the parenthetical term in (64) to be nonzero, while it will force the other term to *be* zero: indeed, if the factor $\chi'_*(t^2x)$ does not vanish then we must have $t^2x\in\mathcal{O}_F^\times$, i.e., $2v(t)+v(x)=0$, contrary to hypothesis.

For every character χ' of $\mathcal{O}_F^\times$, there exists at least one integer $n(\chi')$, such that $v(y)=n(\chi')$ if and only if $\Gamma(y,\omega_\pi\overline{\chi'\chi})-\delta(\omega_\pi\overline{\chi'\chi})\neq 0$. Indeed, if $\omega_\pi\overline{\chi'\chi}$ is nontrivial then there exists *exactly* one such integer $n(\chi')$, given by $n(\chi')=-d-f$, where $\mathfrak{p}^{-d}$ is the kernel of τ and f is the conductor of $\omega_\pi\overline{\chi'\chi}$. If, on the other hand, $\omega_\pi\overline{\chi'\chi}$ is trivial, then for *any* $n(\chi')>-d$ and y such that $v(y)=n(\chi')$, the Gauss sum $\Gamma(y,\omega_\pi\overline{\chi\chi'})$ is simply $\int_{\mathcal{O}_F^\times}\tau(yu)d^*u$; since $\tau(yu)$ is nontrivial for $v(y)>-d$, we see that this Gauss sum has total magnitude strictly less than 1,[27] whence $\Gamma(y,\omega_\pi\overline{\chi\chi'})-\delta(\omega_\pi\overline{\chi\chi'})=\Gamma(y,\omega_\pi\overline{\chi\chi'})-1\neq 0$.

So we must try to choose x, χ', and t so that we may force $2v(t)+x(x)\neq 0$, $v(t)+v(x)=n(\chi')$, and $J_\pi(x,\chi')\neq 0$. But if we we have $2v(t)+x(x)=0$ and $v(t)+v(x)=n(\chi')$, then $v(x)=2n(\chi')$. So we may pick x and χ' such that $J_\pi(x,\chi')\neq 0$ and $v(x)\neq 2n(\chi')$, and then t may be chosen arbitrarily.

What about making $J_\pi(x,\chi')\neq 0$? Here we argue by contradiction. Suppose that we *cannot* find x and χ' such that $J_\pi(x,\chi')\neq 0$ and $v(x)\neq 2n(\chi')$; i.e., $J_\pi(x,\chi')=0 \implies v(x)=2n(\chi')$. Then $J_\pi(x,\chi')\in\mathscr{S}(F^\times)$ for all χ'. But then by (43), we would have $\pi(w)V_*\subset V_*$, which in turn implies $\pi(w)V_*=V_*$, contrary to our initial assumption. Thus

[27] Here we use the triangle inequality: the sum of n complex numbers, each of of whose complex magnitudes is equal to 1 (as is the case for for the image of τ), is strictly less than n, unless all are equal to 1 (in which case it equal to n). Upon taking the associated volume measure in the integral for Γ, this implies the above claim.

there must exist some χ' and t such that (64) does not vanish for some x, and so (64), viewed as a function of x, lies in $\pi(w)V_* \cap V_*(\chi)$, and is nonzero.

□

Now we can prove our earlier claims.

Lemma 5.2. *The space $\pi(w)V_* \cap V_*(\chi)$ has finite codimension in $V_*(\chi)$ for every character χ of $\mathcal{O}_F^\times$.*

Proof. A function in $\xi \in V_*(\chi)$ is uniquely specified by its values $\xi(\varpi^n)$ for each integer n. Moreover, for the resulting function to lie in V_*, all but finitely many $\xi(\varpi^n)$ must be zero. This accounts for all $\xi \in V_*(\chi)$, which we now see is isomorphic, as a vector space, to $\bigoplus_{i=-\infty}^{\infty} \mathbb{C}$.

We will now show that the dimension of the space linear forms on $V_*(\chi)$ that annihilate $\pi(w)V_* \cap V_*(\chi)$ is finite dimensional. Since this is isomorphic to the dual of $V_*(\chi)/[\pi(w)V_* \cap V_*(\chi)]$, this will immediately demonstrate that $\pi(w)V_* \cap V_*(\chi)$ has finite codimension in $V_*(\chi)$.

We recall that the dual of $\bigoplus_{i=-\infty}^{\infty} \mathbb{C}$ is $\prod_{i=-\infty}^{\infty} \mathbb{C}$; explicitly, every linear form λ on $V_*(\chi)$ is given by

$$\lambda(\xi) = \sum_{n\in\mathbb{Z}} \lambda_n \xi(\varpi^n) \tag{66}$$

where λ_n are arbitrarily selected complex constants. (Note that there is no requirement that all but finitely many λ_n equal 0.) Say that λ annihilates $\pi(w)V_* \cap V_*(\chi)$. We now know by Lemma 5.1 that there exists nonzero $\xi \in \pi(w)V_* \cap V_*(\chi)$. Let $\alpha_i := \xi(\varpi^i)$. (Note that all but finitely many α_i are zero.) Then we have:

$$\sum \alpha_i \lambda_i = 0.$$

Since $\pi \begin{pmatrix} \varpi^k & 0 \\ 0 & 1 \end{pmatrix} \xi$ *also* lies in $\pi(w)V_* \cap V_*(\chi)$ for all k, and $\left(\pi \begin{pmatrix} \varpi^k & 0 \\ 0 & 1 \end{pmatrix} \xi\right)(\varpi^i) = \xi(\varpi^{i+k}) = \alpha_{i+k}$, we also have:

$$\sum \alpha_{i+k} \lambda_i = 0 \tag{67}$$

for all $k \in \mathbb{Z}$. Thus the λ_i satisfy a linear recurrence. But only a finite dimensional space of sequences can satisfy a linear recurrence, and so our lemma is proved.

□

Lemma 5.3. *The space $\pi(w)V_*$ contains $V_*(\chi)$ for all but finitely many characters χ of $\mathcal{O}_F^\times$.*

Proof. The proof will rely upon the construction of the nonzero $\xi \in \pi(w)V_* \cap V_*(\chi)$ given by (64) in Lemma 5.1. We let $\chi' = \mathrm{Id}$, and note that if the conductor of χ (which equals the conductor of $\overline{\chi}$) is sufficiently large, then the conductors of $\omega_\pi\overline{\chi}$ and χ will agree. Thus the conductor of $\omega_\pi\overline{\chi}$ will equal the conductor of χ for all but finitely many χ. Say this holds for all $\overline{\chi}$ with conductor $f \geq N$, and suppose that f is so large. The function (64) then becomes

$$\omega_\pi(-x)J_\pi(x,\mathrm{Id})\Gamma(tx,\omega_\pi\overline{\chi}) - \omega_\pi(t^{-1})\mathrm{Id}_*(t^2x)\Gamma(-tx,\overline{\chi}). \tag{68}$$

The second term is 0 unless $2v(t) + v(x) = 0$ and $v(t) + v(x) = -d - f$. But this simply means that $v(t) = d + f$ and $v(x) = -2(d + f)$. Recall that if $v(x) \ll 0$, then $J_\pi(x, \mathrm{Id}) = 0$ – this was a consequence consequence of (39). So increasing N if necessary, we may force $-2(d + f)$ to be so small that, for all $f > N$ and x such that $v(x) = -2(d + f)$, we have $J_\pi(x, \mathrm{Id}) = 0$. Then we may choose t such that $v(t) = d + f$. Then (68), as a function of x, is nonzero if and only if $v(x) = -2(d + f)$. This function lies in $\pi(w)V_* \cap V_*(\chi)$, and is supported on a single class modulo $\mathcal{O}_F^\times$. Thus, we may generate all of $V_*(\chi)$ from this single function (68), as we did in the proof of Lemma 3.7. Hence $\pi(w)V_* \cap V_*(\chi) \supset V_*(\chi)$ for all χ with conductor greater than N – i.e., for all but finitely many characters χ.

□

We are now almost done: the last thing we need to show is the uniqueness of the Kirillov Model.

Proposition 5.4. *The space V' of locally constant complex-valued functions on $F^\times$ satisfying the desiderata of Theorem 3.2 is unique, and the representation π' of $GL_2(F)$ on V', satisfying (5) for a given additive character τ of F, is unique up to a nonzero scalar factor.*

Proof. We return briefly to the ξ' notation of Theorem 3.2. Say that $\xi \mapsto \xi'$ is a mapping from V to some subspace of the space of locally constant function on $F^\times$. Then we have by (5) that

$$\eta = \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi \implies \eta'(x) = \tau(bx)\xi'(ax). \tag{69}$$

We may define the linear form $L(\xi) = \xi'(1)$ on V; we clearly have:

$$\xi'(x) = L\left[\pi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\xi\right]. \tag{70}$$

Thus $\xi \mapsto \xi'$ is uniquely determined by L. But then we know from our commentary after the statement of Theorem 3.2 that L annihilates the space of twisted unipotent averages V_0. Since $\dim(V/V_0) = 1$, we see that L is unique up to a scalar multiple, and the proposition follows.

□

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